Electronic spin states in a magnetic field

It can be shown experimentally (by applying a uniform and static magnetic field $\vec{B}_0$ to hydrogen atoms, for example) that the electron has so-called *spin* quantum states. From a classical (i.e. non-quantum) point of view, spinning is about rotating around its own axis. In this exercise, we focus on the quantum spin states of a single electron which are denoted $|+\rangle$ and $|-\rangle$. The latter correspond to counterclockwise and clockwise rotations around the z axis, respectively, as depicted in Fig. 1. The states $|+\rangle$ and $|-\rangle$ will be used as an orthonormal basis of quantum spin states for the electron.

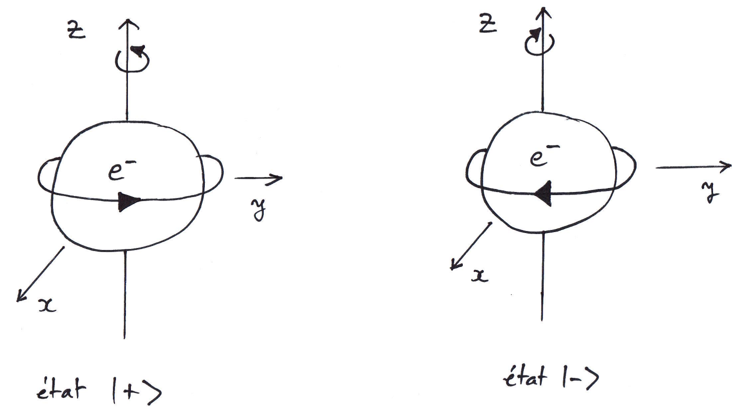

Figure 1: Representation of $|+\rangle$ and $|-\rangle$ spin states [“état” means “state” in French ;-)].

a) If the magnetic field is along the z axis (i.e. $\vec{B}_0 = B_0\,\vec{e}_z$), the matrix representation of the spin Hamiltonian operator in the basis $\{|+\rangle, |-\rangle\}$ reads

$$\left[\hat{H}\right] = \begin{bmatrix} \dfrac{\hbar\omega_0}{2} & 0 \\ 0 & -\dfrac{\hbar\omega_0}{2} \end{bmatrix},$$

where $\omega_0 = \dfrac{eB_0}{m_e}$ is the so-called Larmor frequency [e is the elementary charge of the electron (in absolute value) and m_e its mass]. We assume that, at the initial time $t = 0$, the electron is in the spin state $|+\rangle$. What is the probability to find the electron in the spin state $|-\rangle$ when $t > 0$. Justify your answer without any mathematical derivation.

b) We now assume that the magnetic field is along the x axis (i.e. $\vec{B}_0 = B_0\,\vec{e}_x$). In this case, the matrix representation of the Hamiltonian operator in the basis $\{|+\rangle, |-\rangle\}$ reads

$$\left[\hat{H}\right] = \begin{bmatrix} 0 & \dfrac{\hbar\omega_0}{2} \\ \dfrac{\hbar\omega_0}{2} & 0 \end{bmatrix}.$$

Verify that the eigenvectors of $\hat{H}$ are $|1\rangle = \dfrac{1}{\sqrt{2}}\Big(|+\rangle - |-\rangle\Big)$ and $|2\rangle = \dfrac{1}{\sqrt{2}}\Big(|+\rangle + |-\rangle\Big)$, that they are associated to the energies $-\dfrac{\hbar\omega_0}{2}$ and $\dfrac{\hbar\omega_0}{2}$, respectively, and that they are orthonormal.

c) Let $|\Psi(t)\rangle = C_1(t)|1\rangle + C_2(t)|2\rangle$ be the decomposition of the spin quantum state at time t in the basis of the eigenvectors of $\hat{H}$. Show that $C_1(t) = C_1(0)\,e^{\mathrm{i}\frac{\omega_0}{2}t}$ and $C_2(t) = C_2(0)\,e^{-\mathrm{i}\frac{\omega_0}{2}t}$.

d) We assume that, at time $t = 0$, the electron is in the spin state $|+\rangle$. What are the values of $C_1(0)$ and $C_2(0)$?

e) Show, by calculating $\mathcal{P}_+(t) = |\langle +|\Psi(t)\rangle|^2$, that the spin state of the electron oscillates between $|+\rangle$ and $|-\rangle$ with Larmor's frequency ω_0.

f) Explain why, by referring to Ehrenfest's theorem, the expectation value of the energy $\langle\Psi(t)|\hat{H}|\Psi(t)\rangle$ does not vary with time. What is its value ?

Problème : états de spin de l'électron en présence d'un champ magnétique

a) $\hat{H}|+\rangle = \frac{\hbar\omega_0}{2}|+\rangle \Rightarrow |+\rangle$ est un état propre de l'hamiltonien.

$\boxed{\text{C'est donc un état stationnaire}}$ $\Rightarrow$ si $|\psi(0)\rangle = |+\rangle$ alors $|\psi(t)\rangle$ reste colinéaire à $|+\rangle$ (puisque l'hamiltonien ne dépend pas du temps) et donc orthogonal à $|-\rangle$. La probabilité d'être dans l'état $|-\rangle$, qui vaut $|\langle -|\psi(t)\rangle|^2$, est donc nulle.

b) • Pour $\vec{B} = B_0\vec{e}_x$, $\hat{H}|+\rangle = \frac{\hbar\omega_0}{2}|-\rangle$ et $\hat{H}|-\rangle = \frac{\hbar\omega_0}{2}|+\rangle$

donc $\hat{H}|1\rangle = \frac{1}{\sqrt{2}}\left(\hat{H}|+\rangle - \hat{H}|-\rangle\right) = \frac{1}{\sqrt{2}}\left(\frac{\hbar\omega_0}{2}|-\rangle - \frac{\hbar\omega_0}{2}|+\rangle\right)$

soit $\hat{H}|1\rangle = -\frac{\hbar\omega_0}{2}\underbrace{\left(|+\rangle - |-\rangle\right)\frac{1}{\sqrt{2}}}_{|1\rangle}$

$\rightarrow \boxed{\hat{H}|1\rangle = -\frac{\hbar\omega_0}{2}|1\rangle}$

De même $\hat{H}|2\rangle = \frac{1}{\sqrt{2}}\left(\hat{H}|+\rangle + \hat{H}|-\rangle\right) = \frac{1}{\sqrt{2}}\left(\frac{\hbar\omega_0}{2}|-\rangle + \frac{\hbar\omega_0}{2}|+\rangle\right)$

soit $\boxed{\hat{H}|2\rangle = \frac{\hbar\omega_0}{2}|2\rangle}$

• Comme $\langle +|+\rangle = \langle -|-\rangle = 1$ et $\langle +|-\rangle = 0$

il vient $\langle 1|2\rangle = \frac{1}{2}\left(\langle +|+\rangle + \cancel{\langle +|-\rangle} - \cancel{\langle -|+\rangle} - \langle -|-\rangle\right)$

$\boxed{\langle 1|2\rangle = 0}$

$$\boxed{\begin{aligned}\langle 1|1\rangle &= \tfrac{1}{2}(1+1) = 1\\ \langle 2|2\rangle &= \tfrac{1}{2}(1+1) = 1\end{aligned}}$$

c) $|\psi(t)\rangle = c_1(t)|1\rangle + c_2(t)|2\rangle$

$\downarrow$

vérifie l'équation de Schrödinger dépendante du temps $\hat{H}|\psi(t)\rangle = i\hbar\frac{d}{dt}|\psi(t)\rangle$

soit $c_1(t)\underbrace{\hat{H}|1\rangle}_{-\frac{\hbar\omega_0}{2}|1\rangle} + c_2(t)\underbrace{\hat{H}|2\rangle}_{+\frac{\hbar\omega_0}{2}|2\rangle} = i\hbar\dot{c}_1|1\rangle + i\hbar\dot{c}_2|2\rangle$

d'où $\begin{cases} i\hbar\dot{c}_1 = -\frac{\hbar\omega_0}{2}c_1 \\ i\hbar\dot{c}_2 = \frac{\hbar\omega_0}{2}c_2 \end{cases}$

soit $\dot{c}_1 = \frac{i\omega_0}{2}c_1$ et $\dot{c}_2 = -i\frac{\omega_0}{2}c_2$

$\boxed{c_1(t) = c_1(0)\,e^{\frac{i\omega_0 t}{2}}}$ $\boxed{c_2(t) = c_2(0)\,e^{-\frac{i\omega_0 t}{2}}}$

d) $|\psi(0)\rangle = |+\rangle$ or $|1\rangle = \frac{1}{\sqrt{2}}\left(|+\rangle - |-\rangle\right)$

et $|2\rangle = \frac{1}{\sqrt{2}}\left(|+\rangle + |-\rangle\right)$

donc $|1\rangle + |2\rangle = \frac{2}{\sqrt{2}}|+\rangle$ soit

$|+\rangle = \frac{1}{\sqrt{2}}\left(|1\rangle + |2\rangle\right) \Rightarrow \boxed{c_1(0) = c_2(0) = \frac{1}{\sqrt{2}}}$

e) $\mathcal{P}_+(t) = |\langle + | \psi(t)\rangle|^2$

avec $|\psi(t)\rangle = \frac{1}{\sqrt{2}} e^{i\frac{\omega_0 t}{2}} |1\rangle + \frac{1}{\sqrt{2}} e^{-i\frac{\omega_0 t}{2}} |2\rangle$

Comme $\langle + | 1\rangle = \frac{1}{\sqrt{2}} = \langle + | 2\rangle$

il vient $\langle + | \psi(t)\rangle = \frac{1}{2} e^{i\omega_0 t/2} + \frac{1}{2} e^{-i\omega_0 t/2}$

soit $\langle + | \psi(t)\rangle = \cos(\omega_0 t/2)$

$$\rightarrow \mathcal{P}_+(t) = \cos^2(\omega_0 t/2) = \frac{1}{4}\left(e^{i\omega_0 t/2} + e^{-i\omega_0 t/2}\right)\left(e^{-i\omega_0 t/2} + e^{+i\omega_0 t/2}\right)$$

$$= \frac{1}{4}\Big[2 + \underbrace{e^{i\omega_0 t} + e^{-i\omega_0 t}}_{2\cos\omega_0 t}\Big]$$

$$\boxed{\mathcal{P}_+(t) = \frac{1}{2}(1 + \cos\omega_0 t)}$$

L'électron oscille entre les états $|+\rangle$ et $|-\rangle$ avec une pulsation égale à celle de Larmor.

f) D'après le théorème d'Ehrenfest

$$\frac{d}{dt}\langle \psi(t)|\hat{H}|\psi(t)\rangle = \frac{1}{i\hbar}\langle \psi(t)|\underbrace{[\hat{H},\hat{H}]}_{0}|\psi(t)\rangle$$

donc $\boxed{\langle \psi(t)|\hat{H}|\psi(t)\rangle = \langle \psi(0)|\hat{H}|\psi(0)\rangle} = \langle + |\hat{H}| + \rangle$

$= \frac{\hbar\omega_0}{2}\langle + | - \rangle$

soit $\boxed{\langle \psi(t)|\hat{H}|\psi(t)\rangle = 0}$

Commentaire : on peut le vérifier facilement ici.

$$\hat{H}|\psi(t)\rangle = \frac{1}{\sqrt{2}} e^{i\omega_0 t/2} \underbrace{(\hat{H}|1\rangle)}_{-\frac{\hbar\omega_0}{2}|1\rangle} + \frac{1}{\sqrt{2}} e^{-i\omega_0 t/2} \underbrace{(\hat{H}|2\rangle)}_{\frac{\hbar\omega_0}{2}|2\rangle}$$

d'où

$$\langle \psi(t)|\hat{H}|\psi(t)\rangle = -\frac{\hbar\omega_0}{2\sqrt{2}} e^{i\omega_0 t/2}\langle \psi(t)|1\rangle + \frac{\hbar\omega_0}{2\sqrt{2}} e^{-i\omega_0 t/2}\langle \psi(t)|2\rangle$$

avec $\langle 1|\psi(t)\rangle = \frac{e^{i\omega_0 t/2}}{\sqrt{2}}$

et $\langle 2|\psi(t)\rangle = \frac{e^{-i\omega_0 t/2}}{\sqrt{2}}$

Ainsi

$$\langle \psi(t)|\hat{H}|\psi(t)\rangle = -\frac{\hbar\omega_0}{4} + \frac{\hbar\omega_0}{4} = 0$$