Tutorial: projection of the angular momentum operator along the z axis

Spherical coordinates (r, θ, φ) rather than cartesian coordinates (x, y, z) will be used in this exercise. In addition, we assume that the wavefunction describing a given particle depends only on the angle φ. Thus the inner product of two wavefunctions ψ and χ can be written as $\langle\psi|\chi\rangle = \int_0^{2\pi} d\varphi\, \psi^*(\varphi)\chi(\varphi)$.

1. Prove that the z component of the angular momentum operator $\hat{L}_z \equiv -\mathrm{i}\hbar\frac{\partial}{\partial\varphi}$ is hermitian. [**Hint:** show that $\forall\, \psi, \chi, \quad \langle\psi|\hat{L}_z|\chi\rangle = \langle\chi|\hat{L}_z|\psi\rangle^*$]

2. Prove that the eigenvalues of $\hat{L}_z$ are $m\hbar$, where $m \in \mathbb{Z}$, and that the corresponding normalized eigenfunctions are $\Phi_m(\varphi) = \frac{1}{\sqrt{2\pi}}e^{im\varphi}$.

3. Let us consider that at a given time t_0 the wavefunction describing the particle equals $\psi_0(\varphi) = A\cos^2(\varphi)$ where $A \in \mathbb{R}$. Expand ψ_0 in the basis of the Φ_m functions. Rewrite this expansion with Dirac notations (that is $|\psi_0\rangle = \ldots$) and deduce the value of A for which $|\psi_0\rangle$ is normalized.

4. What values can be measured for the observable L_z at time t_0 ? What are the probabilities ?

5. What are the expectation values for L_z and ${L_z}^2$ at time t_0 ?

Exercise 3 : $\hat{L}_z$ operator

1- $\forall \psi, \chi$ $\quad \langle \psi | \hat{L}_z | \chi \rangle = \int_0^{2\pi} d\varphi \, \psi^*(\varphi) \, (\hat{L}_z \chi)(\varphi) = \int_0^{2\pi} d\varphi \, \psi^*(\varphi) \left(-i\hbar \frac{\partial \chi}{\partial \varphi}\right)$

$$= -i\hbar \int_0^{2\pi} d\varphi \, \psi^* \frac{\partial \chi}{\partial \varphi} = -i\hbar \left(\left[\psi^* \chi\right]_0^{2\pi} - \int_0^{2\pi} d\varphi \, \chi \frac{\partial \psi^*}{\partial \varphi} \right)$$

$\varphi = 0$ and $\varphi = 2\pi$ correspond to the same position in space

$\Rightarrow \psi^*(0)\chi(0) = \psi^*(2\pi)\chi(2\pi)$

$\Rightarrow \langle \psi | \hat{L}_z | \chi \rangle = i\hbar \int_0^{2\pi} d\varphi \, \chi \frac{\partial \psi^*}{\partial \varphi} = \int_0 d\varphi (\hat{L}_z \psi)^* \chi = \langle \chi | \hat{L}_z | \psi \rangle^*$

since $\hat{L}_z \psi = -i\hbar \frac{\partial \psi}{\partial \varphi}$

$\Rightarrow$ $\underline{\hat{L}_z \text{ is hermitian.}}$

2- Let Φ be eigenfunction of $\hat{L}_z$ associated to l_z

$\hat{L}_z \Phi = l_z \Phi \quad (2)$

Since $\hat{L}_z$ is hermitian, $\underline{l_z \in \mathbb{R}}$

Proof :

$\langle \Phi | \hat{L}_z | \Phi \rangle = \langle \Phi | \hat{L}_z | \Phi \rangle^*$ according to question 1

thus $\quad l_z \underbrace{\langle \Phi | \Phi \rangle}_{\neq 0} = l_z^* \langle \Phi | \Phi \rangle$

$\Rightarrow l_z = l_z^* \Rightarrow l_z \in \mathbb{R}.$

(2) $\Leftrightarrow$ $\quad -i\hbar \frac{\partial \Phi}{\partial \varphi} = l_z \Phi$

$\Leftrightarrow \quad \Phi(\varphi) = C e^{-l_z/i\hbar \, \varphi}$

$$\Phi(\varphi=0) = \Phi(\varphi=2\pi) \Rightarrow c\, e^{+\frac{i l_z}{\hbar} 2\pi} = c$$

$$\Rightarrow e^{\frac{2i\pi l_z}{\hbar}} = 1 \Rightarrow \frac{2\pi l_z}{\hbar} = 2\pi m \quad m \in \mathbb{Z}$$

$$\Rightarrow \boxed{l_z = m\hbar \quad m \in \mathbb{Z}}$$

Corresponding eigenfunction $\Phi_m = C_m\, e^{im\varphi}$ (we choose $C_m \in \mathbb{R}$)

Normalization : $\langle \Phi_m | \Phi_m \rangle = 1 = \int_0^{2\pi} C_m^* e^{-im\varphi} \cdot C_m e^{im\varphi} d\varphi = C_m^2 \cdot 2\pi$

$$\Rightarrow \boxed{C_m = \frac{1}{\sqrt{2\pi}}} \Rightarrow \Phi_m = \frac{1}{\sqrt{2\pi}} e^{im\varphi}$$

3- $\psi_0(\varphi) = A\cos^2(\varphi) = \frac{A}{4}\underbrace{(e^{i\varphi}+e^{-i\varphi})^2}_{(e^{2i\varphi}+2+e^{-2i\varphi})}$

$$\psi_0(\varphi) = \frac{A}{4}\left(2\times\sqrt{2\pi}\,\Phi_0 + \sqrt{2\pi}\,\Phi_2 + \sqrt{2\pi}\,\Phi_{-2}\right)$$

$$\boxed{\psi_0(\varphi) = \frac{A\sqrt{2\pi}}{4}\left(2\Phi_0 + \Phi_2 + \Phi_{-2}\right)}$$

Dirac notations :

$$|\psi_0\rangle = \frac{A\sqrt{2\pi}}{4}\left(2|\Phi_0\rangle + |\Phi_2\rangle + |\Phi_{-2}\rangle\right)$$

$$\boxed{\text{if } m \neq l \text{ then } \langle \Phi_m | \Phi_l \rangle = 0}$$

Proof: According to question 1-

$$\langle \Phi_m | \hat{L}_z | \Phi_l \rangle = \langle \Phi_l | \hat{L}_z | \Phi_m \rangle^*$$

$$l\hbar \langle \Phi_m | \Phi_l \rangle = \left(m\hbar \langle \Phi_l | \Phi_m \rangle\right)^*$$

$$= m\hbar \langle \Phi_m | \Phi_l \rangle$$

$$\Rightarrow (l-m)\langle \Phi_m | \Phi_l \rangle = 0$$

$$\Rightarrow \langle \Phi_m | \Phi_l \rangle = 0.$$

Therefore $\langle \psi_0 | \psi_0 \rangle = 1 = \frac{A\sqrt{2\pi}}{4}\left(2\underbrace{\langle \psi_0 | \Phi_0 \rangle}_{A\sqrt{2\pi}\cdot\frac{2}{4}} + \underbrace{\langle \psi_0 | \Phi_2 \rangle}_{\frac{A\sqrt{2\pi}}{4}} + \underbrace{\langle \psi_0 | \Phi_{-2} \rangle}_{\frac{A\sqrt{2\pi}}{4}}\right)$

$$\langle \psi_0 | \psi_0 \rangle = \frac{A^2(2\pi)}{16}(4+1+1) = \frac{\pi A^2}{8}\cdot 6 = \frac{3\pi A^2}{4} = 1$$

$$\Rightarrow A^2 = \frac{4}{3\pi} \Rightarrow \boxed{A = \frac{2}{\sqrt{3\pi}}} \Rightarrow \frac{A\sqrt{2\pi}}{4} = \frac{\sqrt{2\pi}}{4}\cdot\frac{2}{\sqrt{3\pi}} = \frac{1}{\sqrt{6}}$$

Thus $|\psi_0\rangle = \frac{1}{\sqrt{6}}\left(2|\Phi_0\rangle + |\Phi_2\rangle + |\Phi_{-2}\rangle\right)$

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0 can be measured with probability $|\langle\Phi_0|\psi_0\rangle|^2 = \frac{4}{6} = \frac{2}{3}$

$+2\hbar$ ———————— $|\langle\Phi_2|\psi_0\rangle|^2 = \frac{1}{6}$

$-2\hbar$ ———————— $|\langle\Phi_{-2}|\psi_0\rangle|^2 = \frac{1}{6}$

5 - $\langle\hat{L}_z\rangle_{\psi_0} = \langle\psi_0|\hat{L}_z|\psi_0\rangle$ where $\langle\psi_0|\psi_0\rangle = 1$

$$= \langle\psi_0|\Big(\frac{2}{\sqrt{6}}\underbrace{\hat{L}_z|\Phi_0\rangle}_{0} + \frac{1}{\sqrt{6}}\underbrace{\hat{L}_z|\Phi_2\rangle}_{2\hbar|\Phi_2\rangle} + \frac{1}{\sqrt{6}}\underbrace{\hat{L}_z|\Phi_{-2}\rangle}_{-2\hbar|\Phi_{-2}\rangle}\Big)$$

$$\langle\hat{L}_z\rangle_{\psi_0} = \frac{2\hbar}{\sqrt{6}}\underbrace{\langle\psi_0|\Phi_2\rangle}_{\frac{1}{\sqrt{6}}} - \frac{2\hbar}{\sqrt{6}}\underbrace{\langle\psi_0|\Phi_{-2}\rangle}_{\frac{1}{\sqrt{6}}}$$

$$\boxed{\langle\hat{L}_z\rangle_{\psi_0} = 0}$$

Comment: Let A be an observable and $\hat{A}$ its corresponding hermitian operator. We denote $\{|u_i\rangle\}_i$ an orthonormal basis of eigenvectors of $\hat{A}$.

At time t_0 the quantum state $|\psi_0\rangle$ can be written in the basis $\{|u_i\rangle\}_i$ as follows

$|\psi_0\rangle = \sum_i c_i|u_i\rangle$ where $\hat{A}|u_i\rangle = a_i|u_i\rangle$ and $\langle\psi_0|\psi_0\rangle = 1$. The expectation value of $\hat{A}$ for the state $|\psi_0\rangle$ can be written as

$$\langle\hat{A}\rangle_{\psi_0} = \langle\psi_0|\hat{A}|\psi_0\rangle = \sum_i c_i\langle\psi_0|\hat{A}|u_i\rangle$$
$$= \sum_i c_i a_i\langle\psi_0|u_i\rangle$$

Since $\langle u_j|\psi_0\rangle = \sum_i c_i\underbrace{\langle u_j|u_i\rangle}_{\delta_{ij}} = c_j \quad \forall j$

$\langle\psi_0|u_i\rangle = \langle u_i|\psi_0\rangle^* = c_i^*$

Therefore $\langle\hat{A}\rangle_{\psi_0} = \sum_i |c_i|^2 a_i = \boxed{\sum_i \mathcal{P}_i a_i = \langle\hat{A}\rangle_{\psi_0}}$

where $\mathcal{P}_i = |c_i|^2 = |\langle u_i|\psi_0\rangle|^2$ is the probability of being in state $|u_i\rangle$ at time t_0.

We can apply directly this formula for $\hat{L}_z$

$\Rightarrow \langle\hat{L}_z\rangle_{\psi_0} = 0\times\frac{2}{3} + 2\hbar\times\frac{1}{6} - 2\hbar\times\frac{1}{6} = 0$

$$\hat{L}_z^2|\psi_0\rangle = \frac{1}{\sqrt{6}}\Big(\underbrace{2\hat{L}_z^2|\Phi_0\rangle}_{=0} + \underbrace{\hat{L}_z^2|\Phi_2\rangle}_{(2\hbar)^2|\Phi_2\rangle} + \underbrace{\hat{L}_z^2|\Phi_{-2}\rangle}_{(-2\hbar)^2|\Phi_{-2}\rangle}\Big)$$

$$= \frac{4\hbar^2}{\sqrt{6}}\left(|\Phi_2\rangle + |\Phi_{-2}\rangle\right)$$

$$\langle \hat{L}_z^2\rangle_{\psi_0} = \langle\psi_0|\hat{L}_z^2|\psi_0\rangle = \frac{4\hbar^2}{\sqrt{6}}\Big(\underbrace{\langle\psi_0|\Phi_2\rangle}_{\frac{1}{\sqrt{6}}} + \underbrace{\langle\psi_0|\Phi_{-2}\rangle}_{\frac{1}{\sqrt{6}}}\Big)$$

$$= \frac{4\hbar^2}{\sqrt{6}}\cdot\frac{2}{\sqrt{6}} = \frac{8\hbar^2}{6} = \frac{4\hbar^2}{3}$$

Comment: We can therefore calculate the standard deviation for the angular momentum projection L_z at time t_0

$$(\Delta L_z)^2_{\psi_0} = \frac{4}{3}\hbar^2 \Rightarrow (\Delta L_z)_{\psi_0} = \frac{2\hbar}{\sqrt{3}}.$$

$$= \langle\hat{L}_z^2\rangle_{\psi_0} - \langle\hat{L}_z\rangle^2_{\psi_0}$$